Reexamining Special Relativity: Revealing and correcting SR's mathematical inconsistency

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Einstein's Special Relativity transformation equations are the foundation of the modern understanding of space and time. These equations are believed to be mathematically consistent. Here we find that the commonly accepted Special Relativity equations are not mathematically consistent and were created using steps that include subtle, yet significantly important, mathematical errors. Because these findings are mathematical in nature, they can be confirmed independently and are not dependent on any physics terminology associated with Special Relativity. This discovery, and the required correction, has implications on the predictive characteristics of the equations as well as on our theoretical understanding of space and time.

Key Words: Special Relativity, Lorentz Transformations, Newtonian Transformations, Mathematical Inconsistency

1. Introduction

Einstein's Theory of Special Relativity[1] (SR) reconciled the relationship between a moving body and the behavior of light. SR also introduced several paradoxes, the most prominent of which are length contraction and time dilation.[2] Challengers to SR often site logical contradictions in one of the paradoxes. However, since the SR community has already established the meaning of the terms and have explained the paradoxes, such challenges are met with resistance.[3,4,5,6] Furthermore, recent attempts[7,8,9] to challenge or redefine SR have not first identified the root cause of the problem with SR.

This paper differs from previous challenges in that it does not rely on paradoxes nor does it first redefine the commonly accepted meaning or interpretation of the equations or variables. It simply presents the mathematical inconsistencies in each of Einstein's derivations of the SR equations. Many theoretical challengers of SR accept the equations as mathematically correct and have not pursued this path.[10,11] The advantage of a mathematical approach is that it is objectively measurable. Mathematical conclusions are not based on what terms mean; they are based on the adherence to certain mathematical rules. Either the rules are followed or they are not.

If the findings of mathematical errors in Einstein's derivations are found to be correct, SR cannot be subsequently supported on the basis of experimental results. Experimental results can only separate theories into two classes; those that are consistent with the results and those that are not. Importantly, experimental results cannot be used as proof of a mathematically inconsistent theory. An alternative theory, which is mathematically correct and remains consistent with the results, must be found.

2. Revisiting Einstein's SR Transformation Derivations

We begin by illustrating mathematical problems in Einstein's 1905 and 1912 derivations of the SR equations. We will then identify the cause of the problem in his 1905 derivation and correct the equations.

The problem with Einstein's 1905 derivation

This section establishes the mathematical rules used to evaluate Einstein's 1905 manuscript. It then summarizes Einstein's derivation and evaluates, purely on mathematical grounds, the equations against those rules to identify the error. Finally, it begins to address the implications of the mathematical findings.

Algebraic Foundation – The Mathematical Rules

Consider the equation $a = br$. This equation states that a is the product of b multiplied by r, for all values of b and r. In this equation, consider that b is a known constant and that r is the returned value from a function. Notice that r can be expressed as the function $f(m,n)$ such that $r = f(m,n)$, where m and n are function parameters. Thus, the original equation can be rewritten as $a = bf(m,n)$. Since $f(m,n)$ is an equivalent mathematical expression for r, we can say that a is the product of b multiplied by $f(m,n)$.

The key mathematical question is whether or not the equality expressed by the equation $a = bf(m,n)$ changes as a result of the specific arguments passed to the function. This would mean that $a = bf(m,n)$ is true for some subset of m and n, and that $a \neq bf(m,n)$ is true for a different subset of m and n. Of course, the equality is always maintained since a is defined as the value b multiplied by the value returned by the function $f(m,n)$. Since b times $f(m,n)$ always produces the value a, $a = bf(m,n)$ is true for all values m and n. We now express a as a function $h(m,n)$ such that $a = h(m,n)$. Thus we conclude that $h(m,n) = bf(m,n)$ is maintained for all values m and n. The following statements are mathematically equivalent:

$$a = h(m,n) = bf(m,n) = br .$$

These statements enable r to be determined without explicitly using the function $f(m,n)$, if $h(m,n)$ is known. Since we have already established that $h(m,n) = bf(m,n)$ for all values m and n, it follows that $\frac{h(m,n)}{b} = f(m,n)$ for all values m and n. This equation, $\frac{h(m,n)}{b} = f(m,n)$, provides a means to test the validity of a system of equations, h and f, when both functions are provided. Specifically, if $\frac{h(m,n)}{b} = f(m,n)$ is true for one subset of m and n, and $\frac{h(m,n)}{b} \neq f(m,n)$ is true for another subset of m and n, then an error exists in either h or f that must be investigated and corrected.

Einstein's Two Time Equations

Einstein's transformation equations for time and length along the X-axis are $\tau = \dfrac{t - \dfrac{vx}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ and $\xi = \dfrac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}}$, respectively.[12] We note and emphasize that in Einstein's 1905 manuscript the transformations between coordinate systems occurs from (x, y, z, t) to (ξ, η, ζ, τ), rather than to (x', y', z', t') as presented in many textbooks.[13,14] Einstein's use of (ξ, η, ζ, τ) in his 1905 manuscript is equivalent to the use of (x', y', z', t') in his subsequent works. As presented in Fig. 1, our use of the variables (ξ, η, ζ, τ) on the left-hand sides of the transformation equations agree with the equations presented in Einstein's 1905 manuscript.[15]

> Aus dieser und der vorhin gefundenen Relation folgt, daß $\varphi(v) = 1$ sein muß, so daß die gefundenen Transformationsgleichungen übergehen in:
>
> $$\tau = \beta\left(t - \frac{v}{V^2}x\right),$$
> $$\xi = \beta(x - vt),$$
> $$\eta = y,$$
> $$\zeta = z,$$
>
> wobei
>
> $$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}},$$

Source: Annalen der Physik **17**, 891 (1905)

FIG 1. Einstein's final transformation equations. The transformation occurs from (x, y, z, t) to (ξ, η, ζ, τ). Einstein's use of (ξ, η, ζ, τ) is equivalent to his use of (x', y', z', t') in later works.

As presented in Fig. 2, Einstein derives the ξ transformation as the equation[16] $\xi = c\tau$ and builds the $\xi = c\tau$ equation by replacing τ with $\alpha\left(t - \frac{vx'}{c^2 - v^2}\right)$ such that $\xi = c\tau = c\left[\alpha\left(t - \frac{vx'}{c^2 - v^2}\right)\right]$. Einstein continues to build the equation as $\xi = c\tau = c\left[\alpha\left(\frac{x'}{c-v} - \frac{vx'}{c^2 - v^2}\right)\right]$ by replacing t with $\frac{x'}{c-v}$, which, when simplified, produces $\xi = c\tau = \alpha \frac{c^2 x'}{c^2 - v^2}$.

Mit Hilfe dieses Resultates ist es leicht, die Größen ξ, η, ζ zu ermitteln, indem man durch Gleichungen ausdrückt, daß sich das Licht (wie das Prinzip der Konstanz der Lichtgeschwindigkeit in Verbindung mit dem Relativitätsprinzip verlangt) auch im bewegten System gemessen mit der Geschwindigkeit V fortpflanzt. Für einen zur Zeit $\tau = 0$ in Richtung der wachsenden ξ ausgesandten Lichtstrahl gilt:

$$\xi = V\tau,$$

oder

$$\xi = a V\left(t - \frac{v}{V^2 - v^2} x'\right).$$

Nun bewegt sich aber der Lichtstrahl relativ zum Anfangspunkt von k im ruhenden System gemessen mit der Geschwindigkeit $V - v$, so daß gilt:

$$\frac{x'}{V - v} = t.$$

Setzen wir diesen Wert von t in die Gleichung für ξ ein, so erhalten wir:

$$\xi = a \frac{V^2}{V^2 - v^2} x'.$$

Source: Annalen der Physik **17**, 891 (1905)

FIG 2. Einstein begins with the equation $\xi = c\tau$ as the foundation for deriving his transformation equations, resulting in $\xi = c\tau = \alpha \dfrac{c^2 x'}{c^2 - v^2}$.

Finally, Einstein takes the equations $\xi = c\tau = \alpha \dfrac{c^2 x'}{c^2 - v^2}$ and $\tau = \alpha\left(t - \dfrac{vx'}{c^2 - v^2}\right)$, replaces x' with $x - vt$, and multiplies them by $\sqrt{1 - \dfrac{v^2}{c^2}}$, producing the final transformation equation $\xi = c\tau = \dfrac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ and $\tau = \dfrac{t - \dfrac{vx}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$, as previously presented in Fig. 1. The difference between the two derivations is the substitution of $t = \dfrac{x'}{c - v}$ that is made in producing the ξ transformation, but not in producing the τ transformation. In other words, time is represented as $\tau = \alpha\left(\dfrac{x'}{c - v} - \dfrac{vx'}{c^2 - v^2}\right)$ in the ξ transformation and as $\tau = \alpha\left(t - \dfrac{vx'}{c^2 - v^2}\right)$ in the stand-alone time transformation, and equal one another only when $t = \dfrac{x}{c}$.

Evidence of the Mathematical Problem

From the preceding discussion, the Einstein-Lorentz equations can be expressed as functions such that $\xi = h(x,t) = \dfrac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ and $\tau = f(x,t) = \dfrac{t - \dfrac{vx}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$. Since we have shown that Einstein built the ξ transformation as the equation $\xi = c\tau$**,** we must be able to mathematically conclude that

$$\xi = h(x,t) = cf(x,t) = c\tau \ .$$

Also from the preceding discussion, we must be able to show that

$$\frac{\xi}{c} = \frac{h(x,t)}{c} = f(x,t) = \tau$$

to conclude that $\dfrac{h(x,t)}{c} = f(x,t)$ for all values x and t. However, we find that $\dfrac{h(x,t)}{c} \neq f(x,t)$ for the majority of x and t, producing $\dfrac{h(x,t)}{c} = f(x,t)$ only when $t = \dfrac{x}{c}$. Restated, the equations are only valid when $\dfrac{h(x, \frac{x}{c})}{c} = f(x, \frac{x}{c})$ and, as discussed earlier, this finding represents a mathematical error that must be explored and corrected.

Implications

This mathematical analysis suggest that $\xi = c\tau = \dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ is a specific instance of the equation $\xi = c\tau$. It also suggests that the time equation $\tau = \dfrac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ is incorrect. These findings do not agree with the current interpretation of SR, which associates $\dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\dfrac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ with fixed point transformations and $c\tau$ as the equation of a wave front.

Since this paper challenges the current interpretation of SR on mathematical grounds rather than by challenging the meaning of the terms or equations, the mathematical implication that $\dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ is a special case of the equation $c\tau$ is acceptable and must be validated. Support of this conclusion requires 1) confirmation of a mathematical problem in Einstein's other derivations, 2) correction of the equations, and 3) explanation of the revised equations such that they remain consistent with existing experimental evidence.

The problem with Einstein's 1912 derivation

In his previously unpublished 1912 manuscript,[17] Einstein derives the transformation equations using a different technique. As with our previous analysis, we will define the mathematical rules and then evaluate Einstein's derivation using those rules.

Equivalence Relations - The Mathematical Rules

Consider the following two equations

$$\begin{aligned} a &= b \\ c &= d. \end{aligned} \tag{1}$$

We can rearrange these equations to produce

$$\begin{aligned} a - b &= 0 \\ c - d &= 0. \end{aligned} \tag{2}$$

In this rearranged form, because the equations must total zero, we can show that a must have the same value as b, and that c must have the same value as d. In other words, we can show that Equations 1 are equivalent to Equations 2.

We now make an important distinction in how Equations 2 can be associated with one another. Consider the following equation,

$$a - b = c - d\ . \tag{3}$$

We must elaborate on the meaning of Equation 3. Mathematically, the expressions $a-b$ and $c-d$ are equivalent to each other as they satisfy the reflexive, symmetric, and transitive relations. However, the expressions $a-b$, $c-d$, and Equation 3, lack an important aspect of Equations 2, notably that they must equal 0. For example, the following will satisfy Equation 3, but not Equations 1 and 2; $a=10$, $b=5$, $c=20$, and $d=15$.

Notice what happens when we consider the complete statements $a-b=0$ and $c-d=0$. The combined statements $a-b=0$ and $c-d=0$ are not equivalent to the statement $a-b=c-d$, specifically due to a failure to adhere to the rules of the symmetric relation. Thus, the use of $a-b=c-d$ as an equivalent statement for the combined statements $a-b=0$ and $c-d=0$ represents a mathematical error as the constraining information – the fact that the individual equations must equal 0 - is lost.

Evidence of the mathematical problem

Einstein begins his 1912 derivation by stating the equations for two spheres as[18]

$$\begin{aligned}\sqrt{x^2+y^2+z^2} &= ct \\ \sqrt{x'^2+y'^2+z'^2} &= ct',\end{aligned} \tag{4}$$

which can be rewritten as

$$\begin{aligned}x^2+y^2+z^2 &= c^2t^2 \\ x'^2+y'^2+z'^2 &= c^2t'^2.\end{aligned} \tag{5}$$

Since these equations are in the form $a = b$ and $c = d$, Einstein rewrites them as[19]

$$\begin{aligned} x^2 + y^2 + z^2 - c^2t^2 &= 0 \\ x'^2 + y'^2 + z'^2 - c^2t'^2 &= 0 \end{aligned} \tag{6}$$

and states that they "must be equivalent."[20] Einstein then associates[i] the expressions $x^2 + y^2 + z^2 - c^2t^2$ and $x'^2 + y'^2 + z'^2 - c^2t'^2$ to produce[21]

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 . \tag{7}$$

As previously discussed, the expressions $x^2 + y^2 + z^2 - c^2t^2$ and $x'^2 + y'^2 + z'^2 - c^2t'^2$ are equivalent to each other as they satisfy the reflexive, symmetric, and transitive relations. Also discussed previously, $x^2 + y^2 + z^2 - c^2t^2$, $x'^2 + y'^2 + z'^2 - c^2t'^2$, and Equation 7, lack the important constraint from Equations 6, notably that they must equal 0. Now reconsider Equations 6. The combined statements

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \text{ and } x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{8}$$

are not equivalent to the statement

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \tag{9}$$

due to a failure to adhere to the rules of the symmetric relation. Since Equation 9 loses the constraint that the equations must equal 0, Einstein's claim of equivalence is incorrect.

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[i] Einstein writes the equations as $\lambda^2(x^2 + y^2 + z^2 - c^2t^2) = x'^2 + y'^2 + z'^2 - c^2t'^2$. The use of λ^2 does not change the analysis presented in this section. Furthermore, Einstein concludes that λ^2 is 1.

Einstein uses Equation 9 as input into a matrix-based Minkowski calculation to produce the SR transformation equations. While the remainder of Einstein's matrix computation is correct, the error has already occurred. Einstein has actually found a solution for the equations

$$\begin{aligned} x^2 + y^2 + z^2 - c^2t^2 &= g \\ x'^2 + y'^2 + z'^2 - c^2t'^2 &= g \end{aligned} \tag{10}$$

which does not describe a spherical relationship between two coordinate systems.

We conclude that the transformation equations are valid for motion along the X axis when

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \text{ and } x'^2 + y'^2 + z'^2 - c^2t'^2 = 0, \tag{12}$$

which occurs only when $t = \frac{x}{c}$, assuming that y and z equal 0. An associated implication, of course, is that $t' = \frac{x'}{c}$, since y' and z' will equal 0. These findings are consistent with the results of our analysis of Einstein's 1905 manuscript. Einstein use of (x', y', z', t') in this derivation, instead of the Greek variables (ξ, η, ζ, τ), does not change the results of either analysis.

Analysis Confirmation

While the above analysis has focused on the creation of the equation

$$\lambda^2(x^2+y^2+z^2-c^2t^2)=x'^2+y'^2+z'^2-c^2t'^2\,, \tag{13}$$

we can also show that Equation 13 does not follow mathematically from Equations 6. Einstein states that Equations 6 are equivalent to each other, which is mathematically represented as

$$x^2+y^2+z^2-c^2t^2=0 \Leftrightarrow x'^2+y'^2+z'^2-c^2t'^2=0 \tag{14}$$

We will call this statement P. Einstein then mathematically concludes that the above expression produces

$$\lambda^2(x^2+y^2+z^2-c^2t^2)=x'^2+y'^2+z'^2-c^2t'^2 \tag{15}$$

We will call this statement Q. Statement P implies Q, such that any solution for P is also a solution for Q, regardless of the value of λ. Therefore Q can be used as a replacement for P as long as P would have been True. Such an implication does not enable us to pick any combination of (x,y,z,t), but instead we are constrained to only those values that form a sphere. Using Q as a replacement for P for all combinations of (x,y,z,t) represents a mathematical error.

While implication does not allow us to use Q for all combinations of (x,y,z,t) and λ, we need to determine if equivalence will. Although Einstein determines that statement Q is an identity, it does not remove the requirement that P must be equivalent to Q for all values of (x, y, z, t) and λ. This includes the case where λ is 0. In such cases, Q can be True while P is True, indicating that the statements are not equivalent. Using Q as a replacement for P represents a mathematical error because the two statements are not equivalent.

Q does not mathematically follow from P for all values of (x, y, z, t) and λ. Thus, the remainder of Einstein's 1912 derivation is invalidated because Q is incorrectly used to replace P. This finding confirms our earlier analysis.

Einstein's derivation presented in Appendix 1 of his *Relativity* book[22] suffers from the same inconsistencies discussed above.

Implications

The commonly accepted definitions of the SR transformation equations are not mathematically supported since we have found mathematical inconsistencies in each of Einstein's derivations. Therefore, the accepted definition of $\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$ as a fixed point equation remains called into question and this definition cannot be used as a defense against our analysis. This analysis also lends support to the implication established earlier that $\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$ represents a specific instance of ct' (or $c\tau$ using the terminology of his 1905 derivation.).

Correcting the Time Equation

In order to review and effectively illuminate the root cause of the problem in Einstein's 1905 paper, we borrow terminology from the Computer Science discipline. A function is defined as a function name, a set of parameters, and an equation prototype, such that:

$$name(parameters) = equation\ \ prototype\ .$$

As an example, $f(m,n) = m*7+n$, where f is the function name, m and n are the parameters, and $m*7+n$ is the equation prototype.

In a function, the parameters represent placeholders for arguments, which are passed to the function when the function is invoked, or called. For example, when function f is called with actual arguments $m=5$ and $n=4$, we produce the function invocation $f(5,4)$, resulting in the equation $5*7+4$, which evaluates to 39. Functions can also be called symbolically, passing variables as arguments. For example, the function can be called with the variables a and b as actual arguments. In this case, since $m=a$ and $n=b$, invoking the function as $f(a,b)$ produces $a*7+b$ as the equation. We will use the term “instantiation” to indicate the specific equation that results from invoking a function with actual arguments. This leads to an important and significant distinction;

> A function’s equation prototype is not the equation, it is a template for the function’s instantiated equation.

The instantiated function equation must be based on the actual arguments passed to the function. Consider that each of the following equivalent functions;

$$f(m,n) = m*7+n$$
$$f(x_1,x_2) = x_1*7+x_2$$
$$f(s,t) = s*7+t$$

when invoked as $f(a,b)$, each produces the instantiated equation $a*7+b$.

Einstein uses a Partial Differential Equation (PDE) to create the time function τ. Einstein begins by stating that $x' = x - vt$ and then begins to derive the time function τ by noting that $\frac{1}{2}(\tau_0+\tau_2)=\tau_1$, where

$$\tau_0 = \tau(0,0,0,t),$$
$$\tau_1 = \tau(x',0,0,t+\frac{x'}{c-v}),$$
$$\tau_2 = \tau(0,0,0,t+\frac{x'}{c-v}+\frac{x'}{c+v}),$$

resulting in[23]

$$\frac{1}{2}\left[\tau(0,0,0,t)+\tau(0,0,0,t+\frac{x'}{c-v}+\frac{x'}{c+v})\right]=\tau(x',0,0,t+\frac{x'}{c-v}).$$

Einstein treats[24] $t=\tau=0$ (in the above equation), resulting in:

$$\frac{1}{2}\left[\tau(0,0,0,0)+\tau(0,0,0,\frac{x'}{c-v}+\frac{x'}{c+v})\right]=\tau(x',0,0,\frac{x'}{c-v}).$$

This equation is then used to build the PDE as[25]

$$\frac{1}{2}\left(\frac{1}{c-v}+\frac{1}{c+v}\right)\frac{\partial\tau}{\partial t}=\frac{\partial\tau}{\partial x'}+\frac{1}{c-v}\frac{\partial\tau}{\partial t}.$$

Einstein uses this PDE to build the equation prototype, $\alpha\left(t-\frac{vx'}{c^2-v^2}\right)$, such that the τ function is defined as:

$$\tau=\tau(x',y,z,t)=\alpha\left(t-\frac{vx'}{c^2-v^2}\right).$$

This is a more complete function definition than contained in Einstein's statement that "Since τ is a *linear* function, it follows from these equations that $\tau=\alpha\left(t-\frac{v}{c^2-v^2}x'\right)$ where α is [an unknown function]."[26] We emphasize that, since Einstein uses the PDE to produce the function, τ is the function name, x', y, z, and t are the function parameters, and $\alpha\left(t-\frac{vx'}{c^2-v^2}\right)$ is the equation prototype.

As discussed earlier, the function can be defined using different parameter variables. Therefore, each of the following function definitions are equivalent:

$$\tau(a,b,c,d) = \alpha\left(d - \frac{va}{c^2 - v^2}\right),$$
$$\tau(x_1, x_2, x_3, x_4) = \alpha\left(x_4 - \frac{vx_1}{c^2 - v^2}\right),$$
$$\tau(x', y, z, t) = \alpha\left(t - \frac{vx'}{c^2 - v^2}\right)$$

Each of these functions yield the same result when invoked with the same arguments.

We now complete the derivation and show that the time function, when invoked and incorporated into the equation

$$\frac{1}{2}\left[\tau(0,0,0,0) + \tau(0,0,0,\frac{x'}{c-v} + \frac{x'}{c+v})\right] = \tau(x',0,0,\frac{x'}{c-v}),$$

produces

$$\frac{1}{2}\left[\alpha\left(0 - \frac{0v}{c^2 - v^2}\right) + \alpha\left((\frac{x'}{c-v} + \frac{x'}{c+v}) - \frac{0v}{c^2 - v^2}\right)\right] = \alpha\left(\frac{x'}{c-v} - \frac{vx'}{c^2 - v^2}\right),$$

which simplifies to

$$\alpha\frac{x'}{c\left[1 - \frac{v^2}{c^2}\right]} = \alpha\frac{x'}{c\left[1 - \frac{v^2}{c^2}\right]}.$$

Following the replacement of x' with $x - vt$, we find

$$\alpha\frac{x - vt}{c\left[1 - \frac{v^2}{c^2}\right]} = \alpha\frac{x - vt}{c\left[1 - \frac{v^2}{c^2}\right]}.$$

This leads to the conclusion that the instantiated time equation is:

$$\tau_1 = \frac{1}{2}\left(\tau_0 + \tau_2\right) = \tau(x',0,0,\frac{x'}{c-v}) = \alpha\frac{x-vt}{c\left[1-\frac{v^2}{c^2}\right]}.$$

"Since τ is a *linear* function," we conclude that the time and length transformations must be based on the instantiated equation τ_1 and not on the function equation prototype τ.

We have shown that $\tau_1 = \tau(x',0,0,\frac{x'}{c-v}) = \alpha\frac{x-vt}{c\left[1-\frac{v^2}{c^2}\right]}$ and further conclude that

$$\xi = c\tau_1 = c\tau(x',0,0,\frac{x'}{c-v}) = \alpha\frac{x-vt}{\left[1-\frac{v^2}{c^2}\right]}.$$

This leads to a key finding that, due to the use of the same variable names as the parameters and as the actual arguments, Einstein incorrectly simplified the stand-alone τ transformation using the time equation prototype $\alpha\left(t-\frac{vx'}{c^2-v^2}\right)$ instead of instantiating the equation prototype to produce $\alpha\frac{x-vt}{c\left[1-\frac{v^2}{c^2}\right]}$. However, his replacement of t with $\frac{x'}{c-v}$ in deriving the ξ equation resulted in the de facto instantiation of the equation, producing the correct ξ transformation $\alpha\frac{x-vt}{\left[1-\frac{v^2}{c^2}\right]}$. Since time is a function, both transformations must be created using the instantiated time equation.

Notice that the equations $\tau_1 = \alpha \frac{x - vt}{c\left[1 - \frac{v^2}{c^2}\right]}$ and $\xi = \alpha \frac{x - vt}{\left[1 - \frac{v^2}{c^2}\right]}$ have not been multiplied by $\sqrt{1 - \frac{v^2}{c^2}}$ as the time adjustment. We will show later in this paper that this adjustment is not automatic and that the actual value of the adjustment is not fixed.

3. Complete and Incomplete Coordinate Systems

With the mathematical problem in Einstein's time transformation identified and corrected, we can begin to reevaluate and reinterpret the meaning of the corrected equations and their implications. Einstein defined two postulates, stating that:

1. *The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.*
2. *Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body.*[27]

Einstein defined one type of coordinate system,[28] or inertial reference frame, which he associated with the transformation equations. However, we observe that a reference frame moving through a wave medium can have one of two principal behaviors. First, the wave medium can move with respect to the moving body. Second, the wave medium does not move with respect to the moving body. In order to revise the postulates, the definition of a coordinate system needs to be extended. Einstein assumed one type of coordinate system which had two states; stationary and moving.[29,30] This paper retains the states and defines two types of coordinate systems: a Complete Coordinate System and an Incomplete Coordinate System. These systems are defined as:

> <u>A Complete Coordinate System</u> is a coordinate system, K', with respect to coordinate system K, where the underlying phenomenon under observation changes velocity by the same amount as the velocity applied to K'.
>
> <u>An Incomplete Coordinate System</u> is a coordinate system, K', with respect to coordinate system K, where the underlying phenomenon under observation does <u>not</u> change velocity by the same amount[ii] as the velocity applied to K'.

Here we provide an example to illustrate the difference between a Complete and Incomplete Coordinate System[iii]. Consider a train on a straight and flat railroad track. The train represents the moving coordinate system and the ground upon which the tracks are laid, the stationary system. To clearly illustrate the concepts behind the two types of

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[ii] Initially, this paper assumes that the velocity of the wave does not change at all as result of applying velocity to the K' system.

[iii] This example is best read without overlaying it with concepts such as length contraction or time dilation.

coordinate systems, we replace a wave with a surrogate; in this case a jogger.[iv] Begin with the rear of the train at the origin of the track, which is marked along its entire length to indicate distance. The front of the train is at x'. When the train is stationary, a jogger can travel at velocity[v] c, from the rear of the train to its front and return again to the rear. He will travel a total distance of $2x'$. If he is moving at velocity c, the total time for him to make this round trip journey (e.g., one oscillation) is $2\frac{x'}{c}$.

Now consider that the train is moving forward at velocity v. In an Incomplete Coordinate System, the jogger runs along side the train. Of course v has to be less than c or the jogger will never make it to the front of the train. We can again ask the question, how far does the jogger now need to jog in order to reach the front of the train and again return to the rear? The answer to the question, while mathematically more complicated, is found to be $2\frac{x'}{1-\frac{v^2}{c^2}}$. This equation is based on summing the time required to jog to the front of the train, $\frac{x'}{c-v}$, with the time required to jog to the rear, $\frac{x'}{c+v}$, to produce a the total time for his round trip journey as $2\frac{x'}{c\left[1-\frac{v^2}{c^2}\right]}$. The total

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[iv] The author acknowledges that a jogger is not a physical manifestation of a light wave. The intent of these examples is to provide clarity around the concepts of Complete and Incomplete Coordinate Systems.

[v] While c is typically associated with the speed of light, in our example c simply represents the speed of the jogger (e.g., 5 km/h).

distance he jogs is simply this amount of time multiplied by his velocity, c. With these equations, we can ask one final question; how far is half his journey (e.g., one-half the oscillation distance) and how long does that take? The answer to this question is found by dividing these equations by two, producing $\frac{x'}{1-\frac{v^2}{c^2}}$ and $\frac{x'}{c\left[1-\frac{v^2}{c^2}\right]}$ for length and time, respectively.

Can the last question be answered without explicitly knowing the length of the train? Fortunately, the length of the train can be determined if the current position of the front of the train, x, its velocity, v, and how long it's been traveling, time t, are all known. We can use this information and apply the Newtonian equation $x' = x - vt$ to find the original length x', enabling us to answer the question. Alternatively, we can simply replace x' with $x - vt$ in the numerator of the above equations to produce $\frac{x-vt}{1-\frac{v^2}{c^2}}$ and $\frac{x-vt}{c\left[1-\frac{v^2}{c^2}\right]}$.

In a Complete Coordinate System, the person is jogging inside of the train. The person journeys from the rear to the front and back to the rear again. The equations are applied as if the train were stationary with a round trip (e.g., one oscillation) travel length of $2x'$

and a round trip travel time of $2\frac{x'}{c}$. These values can be divided by two to produce x' and $\frac{x'}{c}$, representing one-half the round trip journey length and time, respectively.

With two types of coordinate systems defined, the postulates are updated as follows:

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two *complete* systems of coordinates in uniform translatory motion.
2. Any ray of light moves in a "stationary" or "*complete*" system of coordinates with the determined velocity c, *as defined by the properties of that coordinate system*, whether the ray be emitted by a stationary or by a moving body.

The second postulate is further generalized so that it applies to all waves rather than specifically to rays of light. As such, it can be rewritten as follows, where *c* represents the speed of the wave under observation:

> *2. Any wave moves in a "stationary" or "complete" system of coordinates with the determined velocity c, as defined by the properties of that coordinate system, whether the wave is caused by a stationary or by a moving body.*

Without the distinction between a Complete and an Incomplete Coordinate system, the velocity of the train would have to be restricted to less than that of the jogger in order to satisfy Einstein's original postulates. Both the equations and the postulate would apply to the same type of coordinate system. However, with the postulates revised for Complete and Incomplete Coordinate Systems, the equations $\frac{x-vt}{1-\frac{v^2}{c^2}}$ and $\frac{x-vt}{c\left[1-\frac{v^2}{c^2}\right]}$ apply to the Incomplete Coordinate System and the postulates apply to the Complete Coordinate System. It must be emphasized that in both types of systems the velocity of the train is not limited to the velocity of the jogger.

4. Mathematical Foundations

With a basic foundation of Complete and Incomplete Coordinate Systems established, we now begin to algebraically define the equations for a moving Incomplete Coordinate System by noting that distance is measured in two ways.[31] Distance is determined using non-wave measurements, which Einstein refers to as rigid rods, and are represented in this paper as discrete values or variables (e.g., *x*, *y*, and *z*). Distance is also determined using wave-based measurements, which Einstein refers to as moving rods, which are represented as equations multiplying the speed of the wave by the amount of time required for the wave to travel the necessary distance (e.g., ct_x, ct_y, and ct_z). When measuring distance in a stationary or Complete Coordinate System, the following equations apply such that

$$x = ct_x ,\quad y = ct_y ,\quad \text{and} \quad z = ct_z . \tag{16}$$

We now expand on the train example provided in the previous section. Consider the dimensions of the train as length x along the X axis, length y along the Y axis, and length z along the Z axis. Notice that we use length x instead of x' as was used by Einstein and in our original train example. This is to emphasize the importance of the length along the X axis and recognize that it is not simply a temporary variable.

As in the previous section, begin with the rear of the train at the origin of the track, which is marked along its entire length to indicate distance. The front of the train is at x. When the train is stationary, a jogger can travel at velocity c, from the rear of the train to its front and return again to the rear. He will travel a total distance of $2x$. If he is moving at velocity c, the total time for him to make the round trip journey (e.g., one oscillation) is $2\frac{x}{c}$.

In an Incomplete Coordinate System, the train is moving along the track at velocity v. Performing the same mathematical analysis performed in the previous section, we can determine that the total round trip length and time required for the jogger to complete one oscillation is $2\frac{x}{1-\frac{v^2}{c^2}}$ and $2\frac{x}{c\left[1-\frac{v^2}{c^2}\right]}$, respectively. Finally, one-half the round trip length and time are $\frac{x}{1-\frac{v^2}{c^2}}$ and $\frac{x}{c\left[1-\frac{v^2}{c^2}\right]}$, respectively.

In order to derive the equations for the Y and Z axes, again consider the train stationary positioned with the rear at the origin. The rear of the train has two tail lights, one on the right rear corner and another at the left rear corner. The distance between the two tail lights along the Y axis is y. Now position a new jogger at the left rear tail light. When the train is stationary, the distance that the jogger travels when running from the left tail light to the right tail light and returning to the left tail light (e.g., one oscillation) is $2y$. If he is traveling at velocity c, the total time required to make this journey is $2\frac{y}{c}$.

Now consider that the train is moving forward at velocity v. We again ask the question, in an Incomplete Coordinate System, how far will the jogger need to travel in order to reach the right tail light and return to the left tail light, and how long with it take? The answer for the length, which is found using the Pythagorean Theorem, is $2\frac{y}{\sqrt{1-\frac{v^2}{c^2}}}$. The total time for the jogger to make his round trip journey is $2\frac{y}{c\sqrt{1-\frac{v^2}{c^2}}}$. We can again ask how far the jogger needs to travel to reach half his distance and how long it will take, producing $\frac{y}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\frac{y}{c\sqrt{1-\frac{v^2}{c^2}}}$ for length and time, respectively. In a similar manner, the equations are established for the Z axis.

Mathematically, we establish ξ as the distance that the jogger runs along the X axis, and it is determined by multiplying his velocity, c, by the time it takes to make one-half the journey, τ_ξ. Similar equations are established for length η, which is determined by multiplying velocity c by time τ_η, representing one-half the journey along the Y axis; and along the Z axis using length ζ and time τ_ζ, representing one-half the journey. These equations are mathematically established as

$$\begin{aligned} \xi &= c\tau_\xi, \\ \eta &= c\tau_\eta, \\ \zeta &= c\tau_\zeta. \end{aligned} \tag{17}$$

We will refer to the train as the K' coordinate system and the ground as the K coordinate system. The resulting system of wave-based equations for a moving Incomplete Coordinate System is

$$\xi = \frac{x}{1-\frac{v^2}{c^2}}, \qquad \tau_\xi = \frac{t_x}{1-\frac{v^2}{c^2}},$$

$$\eta = \frac{y}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \tau_\eta = \frac{t_y}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{18}$$

$$\zeta = \frac{z}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \textit{and} \quad \tau_\zeta = \frac{t_z}{\sqrt{1-\frac{v^2}{c^2}}}.$$

In the equations, c is generalized to represent the propagation speed of the wave. In these equations, c (or the speed of the wave) represents a limit on velocity only when a wave relationship is required or desired between points within a moving Incomplete Coordinate System. While the velocity of K' can exceed this limit, oscillations will not occur.

The First Equation Adjustment

The first adjustment to Equations 18 determines the length of K' as x when x is not originally given. As has already been mentioned, when K' is in motion, its position is determined in K by multiplying velocity by the amount of time, t_k, that K' has been in motion, adding this to its original position, x, thus arriving at a value in K. This results in the Newtonian transformation equation $x_k = x + vt_k$. Rearranging this equation we obtain $x = x_k - vt_k$, enabling us to determine the original length x with respect to the K system when x_k and t_k are known. Combining the Newtonian equation $x = x_k - vt_k$

with the wave-based equation $\xi = \frac{x}{1-\frac{v^2}{c^2}}$, we obtain $\xi = \frac{x_k - vt_k}{1-\frac{v^2}{c^2}}$, and since $\xi = c\tau_\xi$,

we also obtain $\tau_\xi = \frac{x_k - vt_k}{c\left[1-\frac{v^2}{c^2}\right]}$.

Since in a moving system, wave-based and non-wave based distances apply and have different meanings, the Newtonian transformation and the Incomplete Coordinate System wave-equations both apply. The non-wave based, commonly referred to as fixed-point, transformations are reestablished as

$$\begin{aligned} &x = x_k - vt_k, \\ &y = y_k, \\ &z = z_k, \\ &\mathit{and} \\ &t = t_k. \end{aligned} \tag{19}$$

The Second Equation Adjustment

The second adjustment assumes that time is kept by measuring the oscillations of the waves traveling along the Y or Z axis. The application of velocity to K' will change the time oscillations since each oscillation takes longer to complete. Einstein normalizes the equations and corrects for the effect of velocity on K' by multiplying the equations by $\sqrt{1-\frac{v^2}{c^2}}$. Since the length equations are multiplied by the same adjustment, it can be inferred that length is determined by counting the time oscillations. Following the application of the second adjustment, we complete the algebraic derivation to arrive at

$$\begin{aligned}
&\xi = \frac{x_k - vt_k}{\sqrt{1-\frac{v^2}{c^2}}}, && \tau_\xi = \frac{x_k - vt_k}{c\sqrt{1-\frac{v^2}{c^2}}}, \\
&\eta = y, && \tau_\eta = t_y, \\
&\zeta = z, \quad \textit{and} && \tau_\zeta = t_z.
\end{aligned} \tag{20}$$

Equations 20 imply that time along the Y and Z axis has not changed, making it easy to overlook the time transformations occurring from t_y to τ_η and from t_z to τ_ζ. While performed mathematically, Einstein does not appear to explicitly acknowledge in his 1905 derivation this time correction nor does he provide reasons for electing to measure time in K' using the wave traveling along the Y or Z axes.

Equation Extensions

The two adjustments are not a requirement in this model. Since time can be measured using timekeeping devices significantly independent of the effects of the given velocity *v* on K', we do not require that a different time be kept in K'. Therefore, we do not automatically multiply the equations by $\sqrt{1-\frac{v^2}{c^2}}$ as a time adjustment, but apply the adjustment when time or length within K' is determined by measuring the wave oscillations within that moving Incomplete Coordinate System, or as needed by application. Furthermore, the adjustment can be as large as multiplying by $1-\frac{v^2}{c^2}$ if time is kept along the X axis rather than along the Y or Z axes. We reintroduce α to account for this adjustment. We also note that x can be replaced by $x_k - vt_k$ when needed by application.

The resulting wave-based equations for an Incomplete Coordinate System are

$$\begin{aligned} \xi &= \alpha \frac{x}{1-\frac{v^2}{c^2}}, & \tau_\xi &= \alpha \frac{t_x}{1-\frac{v^2}{c^2}}, \\ \eta &= \alpha \frac{y}{\sqrt{1-\frac{v^2}{c^2}}}, & \tau_\eta &= \alpha \frac{t_y}{\sqrt{1-\frac{v^2}{c^2}}}, \\ \zeta &= \alpha \frac{z}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \text{and} & \tau_\zeta &= \alpha \frac{t_z}{\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned} \tag{21}$$

Notice, however, that Equations 21 can be generalized to produce wave equations for both Complete and Incomplete Coordinate systems. In order to accomplish this, we introduce the variable μ into the equations. The two extreme cases are determined by μ having a value of 1, representing an Incomplete Coordinate System and 0, representing a Complete Coordinate System. Notice that μ can take on the range of real values between

0 and 1, representing variations of an Incomplete Coordinate System. The equations as applied to Complete and Incomplete Coordinate Systems are revised as[vi]

$$\xi = \alpha \frac{x}{1-\mu\frac{v^2}{c^2}}, \qquad \tau_\xi = \alpha \frac{t_x}{1-\mu\frac{v^2}{c^2}},$$
$$\eta = \alpha \frac{y}{\sqrt{1-\mu\frac{v^2}{c^2}}}, \qquad \tau_\eta = \alpha \frac{t_y}{\sqrt{1-\mu\frac{v^2}{c^2}}}, \tag{22}$$
$$\zeta = \alpha \frac{z}{\sqrt{1-\mu\frac{v^2}{c^2}}}, \quad and \quad \tau_\zeta = \alpha \frac{t_z}{\sqrt{1-\mu\frac{v^2}{c^2}}}.$$

5. Experimental Confirmation and Implications

Because there is a mathematical relationship between this model and the Einstein-Lorentz equations, many experiments that have been used as confirmation of the SR equations will still apply. It is important to recognize that Einstein's SR equations, while mathematically inconsistent, will produce mathematically correct results for the τ_ξ

[vi] This paper has derived the equations for complete and incomplete coordinate systems using c as the variable representing the sustained velocity for the phenomena under observation (e.g., wave). This was done to show the similarity between this model and Einstein's model. And, while we have stated that this variable should be generalized, one cannot help but to continue to associated it with the speed of light. For this reason, this variable should be represented using the variable w. This change will be made to future papers.

equation when $t = \frac{x}{c}$ and the equations are normalized for time by multiplying by $\sqrt{1-\frac{v^2}{c^2}}$. This model will distinguish itself from Einstein's for time calculations as the difference between t and $\frac{x}{c}$ increases. There is no difference between the two models for length along the X axis with the equations normalized by $\sqrt{1-\frac{v^2}{c^2}}$. When the equations are not normalized because the time measurement is performed in the stationary system, the expected difference between the two models of $x\left(\frac{1}{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$ for length along the X axis begins small, but increases as velocity increases.

In agreement with current experimental results, time will appear to run slower in a moving Incomplete Coordinate System when compared to the stationary reference system. However, this model associates the time delay in the moving system with the oscillations traveling farther and taking longer, rather than to unique times within each system. This mathematically consistent theory aligns with experimental results such as Ives-Stilwell.[32] Interestingly, this model suggests that the Michelson and Morley class of experiments should be able to detect absolute movement. This model lends support to the arguments of Ives-Stilwell, Miller, Cahill, Munéra, and others, that the Michelson and

Morley results have been incorrectly analyzed and can be explained using alternative theories. [33,34,35,36,37]

Finally, this model suggests that the properties of the wave medium might be controlled or modified within a Complete Coordinate System. This would enable the wave behavior to appear faster or slower in the Complete Coordinate System than in the stationary reference system. It may be possible to modify current experiments associated with super- and sub-luminal light[38,39,40] to validate the behaviors and equations associated with a Complete Coordinate System.

5. Conclusions

The Relativity in Complete and Incomplete Coordinate Systems model associates no mathematical significance to a light wave over any other type of wave. In addition, the model does not support upper limits on velocity, except when oscillations are required in a moving Incomplete Coordinate System. The model can be used for other types of wave mediums, which may simultaneously coexist with one another. When the equations are generalized to all waves, they clearly support mediums in which the wave speeds are slower than the speed of light as well as mediums in which the wave speeds are faster. The value for velocity, c, is generalized to refer to the speed of the wave through the medium. This model supports the conceptually idea of a yet to be discovered wave medium with properties different than, and with propagation characteristics significantly faster than, those currently associated with EMF. For example, if a quantum wave medium is discovered, it could define a faster-than-light quantum wave velocity that could associate this model with entanglement as observed in quantum mechanics.

The Relativity in Complete and Incomplete Coordinate Systems reestablishes the Newtonian equations for non-wave based, or fixed-point, transformations. The wave-based equations as presented in Equations 22 are defined as a specific instance of their more general equations presented in Equations 17. While the wave-based equations associate the points[vii] (x_k, y, z, t_k) with $(\xi, \eta, \zeta, \tau_\xi)$, there is not a one-to-one relationship between the two points since for any ξ there is one and only one τ_ξ. The implication is that the theoretical predictions of SR, which require a one-to-one relationship between space-time points, will need to be revisited. This model offers an opportunity to revise and extend our understanding of space and time.

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[vii] Using Einstein's notation, these points are written as (x, y, z, t) and $(\xi, \eta, \varsigma, \tau)$, or as (x, y, z, t) and (x', y', z', t').

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