Understanding and Correcting Einstien's 1905 Time Transformation

Steven B Bryant

Primitive Logic, 704 Sansome Street, San Francisco, California, 94111

Steve.Bryant@RelativityChallenge.Com

In Section 3 of Einstein's 1905 Special Relativity paper, he begins the derivation for the fixed-point transformations with the equation $\xi = c\tau$ to produce the transformation equations $\xi = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\tau = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$. This finding is problematic because Einstein's equations, as specified in his 1905 paper, do not agree with the commonly accepted interpretation of $\xi = c\tau$ as a wave-front equation, and with $\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ as fixed-point equations. Second, the final equations violate the mathematical rule that states that *if* $\xi = c\tau$ *then* $\tau = \frac{\xi}{c}$, since generally $\frac{x - vt}{c\sqrt{1 - \frac{v^2}{c^2}}} \neq \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$. Here we show that Einstein's time equation was incorrectly simplified and that the root cause of this mistake is the mistreatment of $\tau = t - \frac{vx'}{c^2 - v^2}$ as an *equation* rather than as a *function*. We will explain the meaning of the partial expression $\frac{vx'}{c^2 - v^2}$, reexamine the meaning of the t variable given that τ is a *function* instead of an *equation*, and explain the equations associated with fixed-point transformations and wave-fronts.

Introduction

Einstein begins his 1905 derivation of the time and length equations with the mathematical statement[1,2,3] $\xi = c\tau$. He performs several algebraic operations to conclude that[4,5,6] $\xi = \dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\tau = \dfrac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$. Mathematically, if $\xi = c\tau$ then $\dfrac{\xi}{c} = \tau$.

However, we find that generally[7,8] $\dfrac{x - vt}{c\sqrt{1 - \frac{v^2}{c^2}}} \neq \dfrac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$. Mathematically, one of these equations must be incorrect. This finding is not only problematic mathematically, but it runs counter to the commonly accepted interpretation of the equations that associates $\xi = c\tau$ with a wave-front, while $\dfrac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $\dfrac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ are associated with fixed point transformations.[9] The scope of this paper is to communicate the correct meaning of Einstein's time *function*, illustrate Einstein's mathematical mistake, and correct the time transformation equation.

Algebraically Deriving the Time Equation

We begin with a concrete example to unambiguously derive the equations for length and time along the X axis. While this example may appear overly simplistic, it provides a common basis for understanding the meaning of Einstein's linear time *function*,

$$\tau = \alpha\left(t - \frac{vx'}{c^2 - v^2}\right).$$

Physical scenario

Consider the following scenario. A bus is parked on a straight road. The road is marked along its entire length, enabling us to measure length. The rear of the bus is located at the origin of the road. Numerically, the origin is labeled zero (0). The front of the bus is located at position x'. Thus, the length of the bus is x'.

There are two joggers, one positioned *inside* of the bus and another positioned *outside* of the bus. Both joggers exhibit and repeat the same behavior; beginning at the rear of the bus, they will jog to the front of the bus, turn around, and return to the rear of the bus. Both joggers travel at a sustained velocity of w. One complete round-trip cycle – rear to front to rear - is referred to as one "oscillation."

Mathematical objective

Given the scenario above, the objective is to determine how long it takes the *outside* jogger to travel the distance of **one-half** an oscillation when the bus is moving forward with velocity v. We must compute both time and distance (a.k.a., length).

Initial mathematical findings

We begin by finding some associated formulas that mathematically describe the scenario. Assuming that the bus is stationary, the time required for the jogger to make the roundtrip journey is $\frac{2x'}{w}$. The total distance traveled by the jogger is simply the amount of time required to make the journey multiplied by his velocity w, resulting in a total distance of $2x'$. One half the total distance and travel time is x' and $\frac{x'}{w}$, respectively. If the jogger is *inside* the bus, rather than running along side, the same equations apply.

As graphically depicted in Fig. 1b, when the bus is moving at velocity v, the time required for the *outside* jogger to run from the rear to the front of the bus is $\frac{x'}{w-v}$. We call this time the **approaching time**. The time required for the *outside* jogger to run from the front of the bus to the rear is $\frac{x'}{w+v}$. We call this time the **receding time**. When the velocity of the bus matches that of the jogger, the approaching time equation is undefined. When the velocity of the bus exceeds that of the jogger, the approaching time equation does not make contextual sense because it yields a negative number. In both cases, the *outside* jogger never reaches the front of the bus. Thus, even though the bus can exceed the velocity of the jogger, the domain of applicability for the equations is when the velocity of the bus is less than that of the jogger.

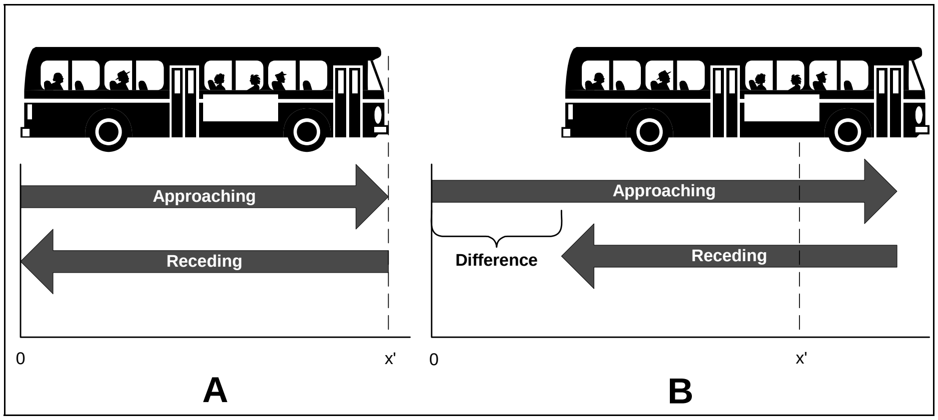

FIG 1. Graphical depiction of time associated with the *outside* jogger with a stationary bus (A) and a moving bus (B). Observe that when the bus is moving, the time required for the *outside* jogger to run from the rear to the front increases. The time required for the *outside* jogger to run from the front to the rear decreases.

Calculating one-half the length and time of one oscillation

We can now answer the questions posed in the *Mathematical Objective* section. There are three ways to calculate length and time for **one-half** an oscillation, given the approaching and receding times. One approach is based on the sum of these two time values, while the other two approaches are based on their difference.

Approach 1 – Using the sum of the time equations

As illustrated in Fig. 1b, the total time required by the *outside* jogger to complete one oscillation is the sum of the approaching time and the receding time, such that $\frac{x'}{w-v}+\frac{x'}{w+v}$, which simplifies to $\frac{2x'}{w\left[1-\frac{v^2}{w^2}\right]}$. Since the total time for one oscillation is

$\dfrac{2x'}{w\left[1-\dfrac{v^2}{w^2}\right]}$, the time for **one-half** of an oscillation is $\dfrac{x'}{w\left[1-\dfrac{v^2}{w^2}\right]}$. When this time is multiplied by velocity w, the total distance for **one-half** an oscillation is $\dfrac{x'}{1-\dfrac{v^2}{w^2}}$.

Approaches 2 and 3 – Using the difference of the time equations

As illustrated in Fig. 1b, the difference between the receding time and the approaching time is found by subtracting the receding time from the approaching time, such that $\dfrac{x'}{w-v}-\dfrac{x'}{w+v}$, or $\dfrac{2vx'}{w^2-v^2}$. One-half of this *difference* is $\dfrac{vx'}{w^2-v^2}$. When one-half of the difference, or $\dfrac{vx'}{w^2-v^2}$, is added to the receding time or is subtracted from the approaching time, the result is **one-half** of the oscillation time. Therefore, as graphically illustrated in Fig. 2, **one-half** the oscillation time can be found as $\dfrac{x'}{w-v}-\dfrac{vx'}{w^2-v^2}$, or as $\dfrac{x'}{w+v}+\dfrac{vx'}{w^2-v^2}$. Both equations, when simplified, produce $\dfrac{x'}{w\left[1-\dfrac{v^2}{w^2}\right]}$ as the time equation. When this time is multiplied by velocity w, the total length for **one-half** an oscillation is found as $\dfrac{x'}{1-\dfrac{v^2}{w^2}}$.

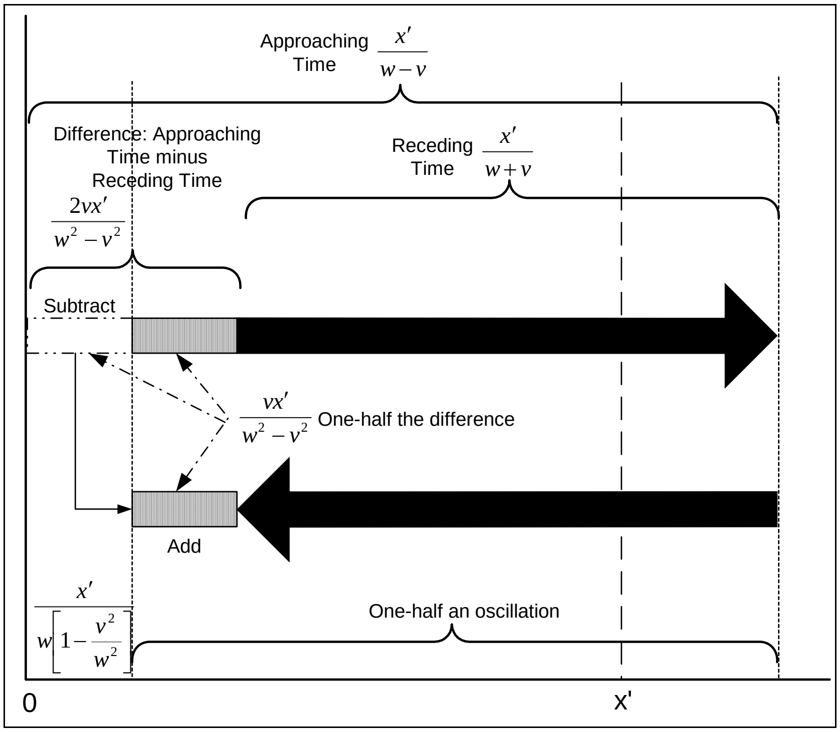

FIG 2. If $\frac{vx'}{w^2-v^2}$ is subtracted from the approaching time, $\frac{x'}{w-v}$, or is added to the receding time, $\frac{x'}{w+v}$, the result is the time required for one-half an oscillation, $\frac{x'}{w\left[1-\frac{v^2}{w^2}\right]}$.

Mathematical extensions

There are two changes we will make to our scenario:

1. Assume that the length of the bus is not initially given, but instead we are given the current position of the bus, x, and the amount of time that the bus has been traveling, t.

2. Change the variable used to represent the sustained velocity of the jogger from w to c.

The first change requires us to compute the length of the bus. We can use the Newtonian equations to compute the original length of the bus as $x' = x - vt$, enabling the use of the previously established time and length equations. Alternatively, we can replace the numerator, resulting in $\frac{x - vt}{w\left[1 - \frac{v^2}{w^2}\right]}$ and $\frac{x - vt}{1 - \frac{v^2}{w^2}}$ for time and length, respectively.

The second change requires us to re-derive the equations as $\frac{x - vt}{c\left[1 - \frac{v^2}{c^2}\right]}$ and $\frac{x - vt}{1 - \frac{v^2}{c^2}}$ for time and length, respectively.

Applying the changes, the three approaches for computing the time equation for one-half an oscillation can be found as

- Approach 1: $\frac{1}{2}\left[\frac{x'}{c - v} + \frac{x'}{c + v}\right] = \frac{x'}{c\left[1 - \frac{v^2}{c^2}\right]}$,
- Approach 2: $\frac{x'}{c - v} - \frac{vx'}{c^2 - v^2} = \frac{x'}{c\left[1 - \frac{v^2}{c^2}\right]}$,
- Approach 3: $\frac{x'}{c + v} + \frac{vx'}{c^2 - v^2} = \frac{x'}{c\left[1 - \frac{v^2}{c^2}\right]}$,

where x' can be optionally replaced with $x - vt$, depending on the given information. With this common foundation, let's consider Einstein's derivation.

Finding the solution using Einstein's Partial Differential Equations

In developing the time transformation, Einstein begins with an unknown *function* τ. As presented in Fig. 3, Einstein invokes this function three times, each time using different arguments, to create a Partial Differential Equation (PDE).

$$\frac{1}{2}\left[\tau\,(0,0,0,t)+\tau\left(0,0,0,\left\{t+\frac{x'}{V-v}+\frac{x'}{V+v}\right\}\right)\right] = \tau\left(x',0,0,t+\frac{x'}{V-v}\right).$$

Hieraus folgt, wenn man x' unendlich klein wählt:

$$\frac{1}{2}\left(\frac{1}{V-v}+\frac{1}{V+v}\right)\frac{\partial\tau}{\partial t}=\frac{\partial\tau}{\partial x'}+\frac{1}{V-v}\frac{\partial\tau}{\partial t},$$

oder

$$\frac{\partial\tau}{\partial x'}+\frac{v}{V^2-v^2}\frac{\partial\tau}{\partial t}=0.$$

Source: Annalen der Physik **17**, 891 (1905)

FIG 3. Einstein's "function invocation" of the unknown τ function, which is used to derive the Partial Deferential Equation.

Einstein uses the PDE to discover the time *function* τ, which is

$\tau(x', y, z, t) = t - \frac{vx'}{c^2 - v^2}$. Notice the similarities between Einstein's *function* body,

$t - \frac{vx'}{c^2 - v^2}$, and the *equation* for finding **one-half** the oscillation time using Approach 2,

$\frac{x'}{c - v} - \frac{vx'}{c^2 - v^2}$. Also notice that Einstein does not include the parameter list in his *function* definition (see Fig. 4), making it easy to confuse the *function* with an *equation*.

> Aus diesen Gleichungen folgt, da τ eine *lineare* Funktion ist:
>
> $$\tau = a\left(t - \frac{v}{V^2 - v^2}x'\right),$$
>
> wobei a eine vorläufig unbekannte Funktion $\varphi(v)$ ist und der Kürze halber angenommen ist, daß im Anfangspunkte von k für $\tau = 0$ $t = 0$ sei.

Source: Annalen der Physik **17**, 891 (1905)

FIG 4. Einstein's *Linear Function* created from the Partial Differential Equation. He does not specify the function τ using a parameter list such as $\tau(x', y, z, t)$. Einstein finds α is 1, enabling us to write the function body as $t - \frac{vx'}{c^2 - v^2}$.

Understanding Einstein's Time Function

Since, as Einstein states, "τ is a linear *function,*" we examine the parameters to understand the function's meaning. Mathematically, the result of a PDE is a function. We will restate Einstein's function using the computer science technique called "pseudo-code" to firmly establish its behavior as a *function*. A function requires the parameters to first be replaced by actual arguments. Consider the following pseudo-code that restates Einstein's *function,* τ, as

$$\tau(length\ x', length\ y, length\ z, approaching\_time\ t)\{\ return\ t - \frac{vx'}{c^2 - v^2}\}$$

Invoking a function takes three steps. First, the parameters in the function definition (e.g., x', y, z, and t) are replaced by the arguments used in the function's invocation (e.g., x', 0, 0, and $\frac{x'}{c - v}$). Second, the replacement is made throughout the function body such that, x' is replaced with x', y is replaced by 0, z is replaced by 0, and t is replaced with $\frac{x'}{c - v}$, creating the equation $\frac{x'}{c - v} - \frac{vx'}{c^2 - v^2}$. Third, this equation is simplified as $\dfrac{x'}{c\left[1-\dfrac{v^2}{c^2}\right]}$, which is the correct time transformation equation that can now be solved. Thus, when the *function* τ is invoked as* $\tau(x',0,0,\dfrac{x'}{c-v})$, as on the right-hand side of the equation expressed in Fig. 3, the result is the equation† $\dfrac{x'}{c-v}-\dfrac{vx'}{c^2-v^2}$, or $\dfrac{x'}{c\left[1-\dfrac{v^2}{c^2}\right]}$. Optionally, since $x'=x-vt$, x' can be replaced with $x-vt$.

Einstein incorrectly treats[10] the *function* $\tau(x',y,z,t)=t-\dfrac{vx'}{c^2-v^2}$ as the *equation* $\tau=t-\dfrac{vx'}{c^2-v^2}$. By first replacing x' with $x-vt$, followed by simplifying the *equation* $\tau=t-\dfrac{v(x-vt)}{c^2-v^2}$, he produces the mathematically inconsistent equation $\tau=\dfrac{t-\dfrac{vx}{c^2}}{1-\dfrac{v^2}{c^2}}$. This is a mathematical error that is detected by performing the validation where, if $\xi=c\tau$ then $\dfrac{\xi}{c}=\tau$. Since t in the linear *function* refers to the approaching time of the *outside*

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* Alternatively, τ could be invoked as $\tau(x-vt,0,0,\dfrac{x-vt}{c-v})$.

† While not derived by Einstein, an alternative PDE based solution can be found as the *function* $\tau(x',y,z,t)=t+\dfrac{vx'}{c^2-v^2}$. This *function* is expressed in pseudo-code as $\tau(length\ x', length\ y, length\ z, receding\_time\ t)\{\ return\ t+\dfrac{vx'}{c^2-v^2}\}$ and is invoked as $\tau(x',0,0,\dfrac{x'}{c+v})$.

jogger, and t in the equation $x' = x - vt$ refers to the amount of time that the bus[‡] has been moving, it is incorrect to confuse them with each another.[§]

Notice that in producing his length equation, Einstein first performs the replacement of t with $\frac{x'}{c-v}$. This de facto function invocation correctly results in the equation $\frac{x'}{c-v} - \frac{vx'}{c^2 - v^2}$. Einstein simplifies the time equation as $\frac{x'}{c\left[1-\frac{v^2}{c^2}\right]}$, which when multiplied by c produces $\frac{x'}{1-\frac{v^2}{c^2}}$ as the correct length equation.

Lastly, Einstein normalizes the final equations by multiplying them by $\sqrt{1-\frac{v^2}{c^2}}$, making it hard to detect the problem.[11] He mathematically performs the normalization without providing supporting text describing his actions.[**] [12]

[‡] The "bus" is a concrete physical example of the conceptual "coordinate system." The *outside* "jogger" represents the phenomenon under observation that oscillates with respect to the moving coordinate system, where its velocity is not governed by the velocity of the moving coordinate system. Einstein's observation with light (or light waves) is only one example of such phenomenon.

[§] A reader with a background in a programming language such as C++ should think in terms of the locally and globally scoped variables with the same name (e.g., the global t variable has one meaning, while the local t function variable has another meaning within the context of the function).

[**] While Einstein does not explain this action in his manuscript, it is apparent that his intent was to normalize the y' and z' equations so that they do not change as a result of the transformations.

Equations for the Y and Z axes

The equations associated with the Y and Z axes can be readily found using the Pythagorean Theorem. The equations for length and time along the Y axis are denoted using the variables η and τ_η, respectively. Similarly, the equations for length and time along the Z axis are denoted using the variables ζ and τ_ζ, respectively. To avoid confusion with the other time variables, we will denote time along the X axis as τ_ξ.

Thus, we arrive at

$$\xi = \frac{x'}{1-\frac{v^2}{c^2}}, \qquad \tau_\xi = \frac{x'}{c\left[1-\frac{v^2}{c^2}\right]},$$

$$\eta = \frac{y}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \tau_\eta = \frac{y}{c\sqrt{1-\frac{v^2}{c^2}}},$$

$$\zeta = \frac{z}{\sqrt{1-\frac{v^2}{c^2}}}, \; and \quad \tau_\zeta = \frac{z}{c\sqrt{1-\frac{v^2}{c^2}}}.$$

Eq. 1

as the complete system of equations. Table I summarizes the meaning of these equations.

Table I. Comparison of the Transformation Equations

	Newton	**Einstein**	**Bryant**
Fixed Point Equations	$x' = x - vt$ $y' = y$ $z' = z$ $t' = t$ • Can apply to a point and a length. • Typically interpreted as applying to a point.	$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ $y' = y$ $z' = z$ $t' = \frac{t - \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$ • Commonly interpreted as applying to a point. • Incorrect time equation. • Equations have been normalized by multiplying each by $\sqrt{1 - \frac{v^2}{c^2}}$.	$x' = x - vt$ $y' = y$ $z' = z$ $t' = t$ • Can apply to a point and a length • Reestablishes the Newtonian equations. • Applies to a length when used with the One-Half Oscillation Equations.
Wave Front Equations	Not Applicable.	$\xi = c\tau$	$\xi = c\tau_\xi$, $\eta = c\tau_\eta$, and $\zeta = c\tau_\zeta$
One-Half Oscillation Equations	Not Applicable.	Not Applicable.	$\xi = \frac{x'}{1 - \frac{v^2}{c^2}}$, $\tau_\xi = \frac{x'}{c\left[1 - \frac{v^2}{c^2}\right]}$ $\eta = \frac{y}{\sqrt{1 - \frac{v^2}{c^2}}}$, $\tau_\eta = \frac{y}{c\sqrt{1 - \frac{v^2}{c^2}}}$ $\zeta = \frac{z}{\sqrt{1 - \frac{v^2}{c^2}}}$, $\tau_\zeta = \frac{z}{c\sqrt{1 - \frac{v^2}{c^2}}}$ • Equations can be normalized. • x' can be replaced with $x - vt$. • Represents a specific instance of the wave front equations. • Defines length and time equations for X, Y, and Z axes.

Conclusion

This paper has explained the root cause of Einstein's incorrect time transformation equation. The cause of the incorrect time transformation was Einstein's mistreatment of the linear time *function* as an equation. By unambiguously explaining the meaning of the terms t and $\frac{vx'}{c^2 - v^2}$ in the *function* $\tau(x', y, z, t) = \alpha(t - \frac{vx'}{c^2 - v^2})$, we were able to explain the true meaning of the time *function* and find the correct mathematical equation.

The correction of the time equation along with the revised understanding of the equations requires us to revisit our understanding of space and time. Newton introduced fixed-point transformations. Einstein extended Newton's model by introducing wave front equations, modifying Newton's fixed-point equations in the process. This model 1) modifies Einstein's fixed-point equations, reestablishing Newton's equations, 2) corrects Einstein's incorrect time equation, and 3) introduces equations for **one-half** oscillations. These findings require Einstein's postulates to be extended for Complete and Incomplete Coordinate systems, which describe the behaviors of the *inside* and *outside* "joggers," respectively. This important extension is beyond the scope of this paper, but is established in *Reexamining Special Relativity.*[13]

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[1] S. Bryant, *Reexamining Special Relativity: Revealing and Correcting SR's Mathematical Inconsistencies* (unpublished) (2005).

[2] S. Bryant, *Communicating Special Relativity Theory's Mathematical Inconsistencies*, (submitted to Galilean Electrodynamics April 2005).

[3] A. Einstein, Annalen der Physik **17**, 891 (1905), (German version obtained in public domain at <http://www.wiley-vch.de/berlin/journals/adp/890_921.pdf> and the English translation obtained in public domain from <http://www.fourmilab.ch/etexts/einstein/specrel/www/>.

[4] S Bryant, 2005, op. cit. (see reference 1).

[5] S Bryant, 2005, op. cit. (see reference 2).

[6] A Einstein, 1905, op. cit. (see reference 3).

[7] S Bryant, 2005, op. cit. (see reference 1).

[8] S Bryant, 2005, op. cit. (see reference 2).

[9] Personal e-mail communication with Dr. Tom Van Flandern, Meta Research.

[10] S Bryant, 2005, op. cit. (see reference 1).

[11] Ibid.

[12] Ibid.

[13] Ibid.